Consumption and Portfolio Rules with Stochastic Hyperbolic Discounting*

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Abstract

We extend the classic Merton (1969, 1971) problem that investigates the joint consumption-savings and portfolio-selection problem under capital risk by assuming sophisticated but time-inconsistent agents. We introduce stochastic hyperbolic preferences as in Harris and Laibson (2008) and find closed-form solutions for Merton's optimal consumption and portfolio selection problem in continuous time. We find that the portfolio rule remains identical to the time-consistent solution with power utility and no borrowing constraints. However, the marginal propensity to consume out of wealth is unambiguously *greater* than the time-consistent, exponential case and, importantly, it is also *more responsive* to changes in risk. These results suggest that hyperbolic discounting with sophisticated agents offers promise for contributing to explaining important aspects of asset market data.

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1 Introduction

In experiments with humans and animals, subjects often exhibit a reversal of preferences when choosing between a smaller, earlier reward and an alternative larger, later reward. The smaller, earlier reward is often preferred when both rewards are near, while the larger, later reward is preferred as they draw more distant.[1] The persistence and robustness of these dynamic inconsistencies has led some economists and psychologists to think "that the problem may not come from some extraordinary condition that impairs the normal operation of intentionality, but rather from the process by which all people, perhaps all organisms, evaluate future goals" (Ainslie and Haslam, 1992, p. 58). Recent neurological evidence supports this view (McClure et al, 2004).

Dynamically inconsistent behavior was first analyzed by David Hume, Adam Smith, and later by Eugene Böhm-Bawerk, William S. Jevons, Alfred Marshall, Wilfredo Pareto, and others in their discussion of passions, sentiments and intertemporal trade-offs. However, it was not until Strotz (1955) that it was first formalized analytically. This first formalization approximates the temporal discount function by a function that discounts more heavily than the exponential function for events in the near future, but less heavily for events in the distant future. Beginning with the work of Laibson (1994, 1997), during the last decade and a half an important body of literature has studied the kind of behavior that rational economic agents with hyperbolic discount functions may exhibit.[2] In particular, in order to attain their goals, individuals may prefer to *restrict* their own future choices. The

[1] See Herrnstein (1997) and other references therein.

[2] This discount function has been used to model a wide range of behavior, including consumption behavior, contracts, addiction, and others. See Harris and Laibson (2001), O'Donoghue and Rabin (1999), DellaVigna and Malmendier (2004), and other references therein.

most apparent way for an individual to forestall his change in preferences is to adopt some type of commitment device.

Gul and Pesendorfer (2001) propose an alternative approach to incorporate the evidence on preferences for commitment. They suggest that *temptation* rather than a *preference change* per se (that is, rather than "dynamic inconsistency") may be the cause of these preferences.[3] Gul and Pesendorfer (2004) extend the analysis to an infinite horizon in an attempt to capture the experimental evidence with tractable, dynamically *consistent* preferences.

A particularly important aspect of this research is the extent to which dynamic inconsistency, temptation, and self-control problems may help us understand individuals' consumption-saving decisions, as well as their decisions to allocate savings among available investment opportunities. Understanding these decisions is, after all, at the heart of a large literature spanning the last few decades on consumption, savings, asset pricing, macroeconomics and other areas. Households are both consumers and investors, and their decisions reflect these dual roles. As consumer, a household chooses how much of its income and wealth to allocate to current consumption, and thereby how much to save for future consumption including bequests. As investor, the household solves the portfolio-selection problem to determine the allocations of its savings among the available investment opportunities. As the modern finance literature emphasizes, the optimal consumption-saving and portfolio-selection decisions typically *cannot* be made independent of each other (see Merton 1969,1971).

The purpose of this paper is to study the effects of dynamic *inconsistency*

[3] They develop a two-period axiomatic model where an ex ante inferior choice may tempt the individual in the second period. Individuals have preferences over sets of alternatives that represent the second period choices. Their representation of preferences identifies the individual's commitment ranking, temptation ranking, and costs of self-control. Moreover, their model yields both different behavioral and normative implications than the change in preferences captured by the hyperbolic discounting approach.

on the joint consumption-saving and portfolio-selection problem. Interestingly, despite the fact that the consumption-saving problem has received substantial attention in the literature on dynamically inconsistent preferences, the consumption-saving *and* portfolio-selection problem has received virtually no attention.[4] In particular, we will examine the implications of a hyperbolic discount function for the lifetime consumption-saving and portfolio-choice problem of an individual household in a continuous-time setting. The analysis may be considered relevant for the following reasons:

First, it is important to evaluate whether emotions and self-control play a role in considerations involving time and *risk* preferences, and hence in intertemporal consumption, saving, and portfolio decisions and in asset prices.[5] Hirshleifer (2001), for example, surveys and assesses the theory and evidence regarding investor psychology as a determinant of asset prices, and considers that "this issue is at the heart of a grand debate in finance spanning the last two decades" (p. 1552). Gul and Pesendorfer (2004) have shown that their dynamically *consistent* preferences do have relevant implications for these decisions. In particular, increasing the agents preference for commitment while keeping self-control constant increases the size of the equity premium.[6] Yet, the extent to which dynamically *inconsistent* preferences have relevant implications for consumption-saving allocations, portfolio choices and asset prices remains unaddressed in the literature.

Second, as emphasized in the finance literature and indicated above,

[4] There is an important amount of work on intertemporal consumption-savings decisions (e.g., Laibson (1994, 1997), Krusell and Smith (2003), Harris and Laibson (2001)). Luttmer and Mariotti (2003) is, to the best of our knowledge, the only paper that also considers households' portfolio decisions. However, they do not study the response of consumption and prices to changes in risk.

[5] Halevy (2008) offers some experimental evidence of the interplay between risk and time preferences.

[6] Similarly, Krusell et al (2002) elaborate on the Gul-Pesendorfer framework which they use to interpret wealth and asset pricing data.

consumption-saving and portfolio-selection decisions typically cannot be made independently of each other. In this sense, the available evidence from the consumption-saving problem need not be sufficient to provide a complete understanding of these decisions.

Third, these joint decisions have been subject to a great deal of theoretical and empirical scrutiny in the consumption-based asset pricing literature under exponential discounting, in particular in the extensive literature on the equity premium puzzle and the excess volatility puzzle.[7] As a result of these efforts, empirical evidence is readily available to evaluate the implications of a hyperbolic discounting structure for observed market data on consumption and security returns.

Lastly, during the last two decades several attempts have been made in the literature to try to resolve the equity premium and other asset pricing puzzles by departing in increasingly complicated ways from the tractable framework of a representative agent, time-additive isoelastic preferences, and complete frictionless markets. In this paper we maintain the standard tractable framework with preferences defined only over consumption.

More precisely, the paper examines the intertemporal consumption and portfolio choice problem of an investor with dynamically inconsistent preferences in a stochastic dynamic programming setting. We consider this setting because it offers valuable advantages. In particular, the use of continuous-time methods has become an integral part of financial economics, and has produced models with a rich variety of testable implications (see Sundaresan (2000) for a review). The adoption of a continuous-time model, in addition, offers a crucial advantage over much of the literature, which has mostly adopted a discrete-time discount function $\left\{1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\right\}$, $\beta \in (0,1)$, $\delta \in (0,1)$, to model the gap between a high short-run discount rate and

[7] See, for instance, Kocherlakota (1996), Campbell (2000) and Mehra (2008) for reviews.

a low long-run discount rate. As Harris and Laibson (2001) and other authors have noted, a recurrent problem that plagues most applications of the discrete-time discount function employed in the literature is that strategic interactions among intrapersonal selves often generates counterfactual policy functions where consumption functions are not globally monotonic in wealth, and may even drop discontinuously at a countable number of points.[8] Moreover, hyperbolic Markov equilibria are not unique in deterministic discrete-time settings (Krusell and Smith, 2003).

These problems can be avoided in a continuous-time setting. Our approach is motivated by Harris and Laibson (2008) Instantaneous Gratification (IG) model, which is based on a quasi-hyperbolic stochastic discount function.[9] The IG model is dynamically inconsistent and, while it captures the qualitative properties of the discrete-time $\beta - \delta$ model, it resolves the pathologies of multiplicity of equilibria and non-monotonicity of the consumption function that have flawed previous theoretical advances in the literature of time-inconsistent preferences.[10] Interestingly, it turns out that our model will yield closed-form solutions for the optimal consumption and portfolio rules that makes them readily comparable to those results obtained under a constant rate of time preference as in Merton (1969, 1971).

[8] These pathologies often occur only in a limited region of the parameter space which, as Harris and Laibson (2008) indicate, typically includes defensible calibrations of the parameters. O'Donoghue and Rabin (1999) note that these pathologies arise only to the extent that individuals are sophisticated, i.e. they are aware of their dynamic inconsistencies. It would be needed to assume that individuals are *completely* naive about their dynamic inconsistency problem; otherwise, the pathologies would be reinstated.

[9] They show the existence and uniqueness of a hyperbolic equilibrium, and the equilibrium consumption function is continuous and monotonic in wealth.

[10] See Harris and Laibson (2008, Section 5).

2 A Lifetime Consumption-Portfolio Problem with Hyperbolic Discounting

We study the classical Merton (1969, 1971) intertemporal consumption-saving and portfolio-selection problem with hyperbolic discounting preferences rather than with exponential discounting ones. An individual is due to make consumption and portfolio decisions that maximizes his discounted lifetime utility of consumption. We assume infinite lifetime, complete markets and no borrowing constraints.

Since our setup is an extension of Harris and Laibson (2008),[11] we attempt to adopt their notation whenever possible. The individual's wealth x_t at any time t can be invested in two assets: a riskless bond with value B_t and a risky asset for an amount N_tP_t, where N_t is the quantity held the of risky-asset and P_t is its price at time t; in particular,

$$x_t = N_t P_t + B_t.$$

While the risk-free asset earns a constant rate of return r continuously, the price P_t follows a geometric Brownian motion with drift μ and diffusion parameter σ, where we assume away a dividend process. Specifically,

$$dB_t = rB_t dt,$$

$$dP_t = \mu P_t dt + \sigma P_t dz_t, \tag{1}$$

where z_t is a standard Wiener process. The change in the individual's wealth during a period of infinitesimal duration dt is determined by the investment proceeds minus consumption $c_t dt$:

$$dx_t = \left[\mu \theta_t x_t + (1 - \theta_t)\, r x_t - c_t\right] dt + \sigma \theta x_t dz_t, \tag{2}$$

[11] Harris and Laibson (2008) consider a setup with one (risky) asset only and impose credit constraints.

where θ_t is the proportion of wealth invested in the risky asset at time t.

Following Harris and Laibson's (2008) quasi-hyperbolic setup, the consumer-investor seeks to maximize his expected lifetime discounted utility of consumption:

$$E_t\left[\int_t^{t+\tau_t} \delta^{(s-t)} u\left(c\left(x_s\right)\right) ds + \int_{t+\tau_t}^{\infty} \beta\delta^{(s-t)} u\left(c\left(x_s\right)\right) ds\right], \tag{3}$$

where $\beta \in (0, 1]$ and $\delta \in (0, 1]$. The discount function decays exponentially at rate $\gamma = -\ln\delta$ up to time $t + \tau_t$, drops discontinuously at $t + \tau_t$ to a fraction β of its level just prior to $t + \tau_t$, and thereafter decays exponentially at a rate $\gamma = -\ln\delta$. The arrival of the "future" is stochastic. In particular, τ_t is distributed exponentially with parameter $\lambda \in [0, \infty)$, which effectively smooths the discount factor and avoids having a kinked or discontinuous discount factor.[12] The IG model in Harris and Laibson (2008) corresponds to the limit $\lambda \to \infty$.

As is well known, a closed-form solution can be found for the case of constant relative risk aversion (CRRA) utility when discounting is exponential (Merton 1969, 1971). For this reason, we consider the utility flow

$$u\left(c\right) = \frac{c^{1-b}}{1-b}, \tag{4}$$

where $b > 0$ is the risk aversion parameter.

Lifetime utility is maximized subject to the budget equation (2) and initial wealth $x_t > 0$. Markets are perfect and there are no taxes, transaction costs, trading restrictions or other impediments to trade. Also, there are no commitment mechanisms. In other words, the introduction of hyperbolic

[12] Alternatively, as noted by Harris and Laibson (2003), the value function can be formulated as $w\left(x_t\right) = E_t\left[\int_t^{\infty} D_\lambda\left(t,s\right) u\left(c\left(x_s\right)\right) ds\right]$ where the discount factor $D_\lambda\left(t,s\right)$ is stochastic and equal to

$$D_\lambda\left(t,s\right) = \begin{cases} e^{-\gamma(s-t)} & \text{wp} \quad e^{-\lambda(s-t)} \quad \Longleftrightarrow \text{ if } \quad s-t \leq \tau_t \\ \beta \cdot e^{-\gamma(s-t)} & \text{wp} \quad 1-e^{-\lambda(s-t)} \quad \Longleftrightarrow \text{ if } \quad s-t > \tau_t. \end{cases}$$

discounting preference is the only difference with respect to the classic formulation of the problem in the literature. As Merton (1969, 1971) shows, this problem can be solved in closed form for optimal consumption and portfolio rules under exponential discounting. We will show next that an explicit solution also exists for the general stochastic hyperbolic-discounting case.[13]

We consider the continuous-time generalization introduced by Harris and Laibson (2008) for two reasons:

i. In order to solve for the Markov perfect Nash equilibrium of the intrapersonal game induced by the hyperbolic discounting structure we use the equivalence result in Barro (1999), Laibson (1997), and Luttmer and Mariotti (2003). They show that in the special case of CRRA utility with no liquidity constraints and no commitment devices, the equilibrium of the intrapersonal game exists and is *observationally equivalent* to a dynamically consistent optimization problem that shares the same instantaneous utility function and equilibrium policy functions but with a different, higher long-run discount rate.[14] This allows us to solve the model using a Bellman System.

ii. The IG model is an important step forward in the treatment of hyperbolic discounting preferences since many of the pathologies of the discrete-time hyperbolic models are eliminated in the continuous time case when $\lambda \to \infty$ as will be discussed later.

The problem for the consumer-investor is to maximize the current-value

[13]Note that the exponential-discounting setup corresponds to the particular hyperbolic-discounting case with $\lambda \to 0$ or $\beta = 0$.

[14]The observationally equivalence critique does not apply to Harris and Laibson (2008) thanks to the liquidity-constraints assumption. The IG model with liquidity-constrained individuals has the same value function of a dynamically consistent optimization problem with the difference that the utility function is wealth contingent. In our problem, however, there are no liquidity constraints.

function

$$w(x_t) = E_t \left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds + e^{-\gamma\tau_t} \beta v(x_{t+\tau_t}) \right], \tag{5}$$

where $\gamma \in (0, +\infty)$ is the discount rate and $v(x_{t+\tau_t})$ is the continuation-value defined as

$$v(x_\zeta) \equiv \int_\zeta^\infty e^{-\gamma(s-\zeta)} u(\tilde{c}(x_s))\, ds,$$

which discounts utility flows exponentially, and where $\tilde{c}$ stands for the consumption levels *optimally* chosen by future selves. The maximization problem is subject to the budget equation (2) and the constraints $c_s \geq 0$ and $x_s \geq 0$, $\forall\ s \geq t$, given an initial wealth endowment $x_t > 0$.

Assumptions

We impose the following set of assumptions:

A1.	$b > 1 - \beta$	(feasibility condition),
A2.	$\gamma > (1-b)\left(\bar{\mu} - \frac{1}{2}b\bar{\sigma}^2\right)$	(integrability condition),
A3.	$\lim_{\zeta\to\infty} E_t \{\exp(-\gamma t) v(x_\zeta)\} = 0,$	(transversality condition),

where $\bar{\mu}$ and $\bar{\sigma}^2$ in **A2** are the mean and variance of the optimal portfolio return rate, as will be characterized below. **A1** and **A2** ensure that the problem is well defined in the sense that the program has a finite solution. As discussed in Harris and Laibson (2008, Section 5.1), in practice, these two assumptions are always satisfied at the empirical estimates of the coefficients b, β and γ typically obtained in the literature. **A3** is a convergence condition for integral (5).

In what follows, for notational convenience we dispense with the time subscripts unless it becomes otherwise necessary.

The current-value function (5) can be written recursively as

$$w(x) = u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[e^{-\gamma dt} \beta v(x+dx)] \tag{6}$$

which satisfies the Bellman equation

$$\gamma w(x) = u(c(x)) + \frac{E[dw(x)]}{dt} + \lambda[\beta v(x) - w(x)], \tag{7}$$

where the second term in the right-hand side can be derived applying Ito's Lemma to (2) :

$$\frac{E[dw(x)]}{dt} = (rx + (\mu - r)\theta x - c)\ \ w'(x) + \frac{1}{2}\sigma^2\theta^2 x^2 w''(x).$$

As noted by Harris and Laibson (2008), the term $\lambda(\beta v(x) - w(x))$ in (6) reflects the hazard rate λ of making the transition from the "presen" to the"future," at which point the continuation value $v(x)$ begins. The intertemporal discount function that applies to the utility flows pertaining to the "future" is a fraction β of the function that prevails in the "present.". Of course, there is no transition effect if $\beta = 1$. The intuition is that when $\beta = 1$ there is no difference in how present utilities and in future utilities are discounted, in which case we would have obtained the classic expression of the time-consistent Bellman equation. The same would be true if the transition probability from "present" to "future" were nil, i.e. if $\lambda = 0$.

Let $\{c^*, \theta^*\}$ denote the optimal policy set of the problem defined as

$$\{c^*(x), \theta^*(x)\} = \arg\max_{c,\theta} \{w(x)\}$$

Recalling the Bellman equation (7), we can write

$$\begin{aligned}\{c^*(x), \theta^*(x)\} \quad = \quad & \arg\max_{c,\ \theta} \{u(c) + (rx + (\mu - r)\theta x - c)\, w'(x) \\ & + \frac{1}{2}\sigma^2\theta^2 x^2 w''(x) + \lambda(\beta v(x) - w(x))\},\end{aligned} \tag{8}$$

or, suppressing the terms that do not contain the controls,

$$\{c^*(x), \theta^*(x)\} = \arg\max_{c,\theta}\{u(c) + ((\mu - r)\theta x - c) w'(x) + \frac{1}{2}\sigma^2\theta^2 x^2 w''(x)\}.$$

The unique interior optimum from the first order conditions determines the optimal consumption and portfolio policies. In particular, such conditions are

$$0 = (\mu - r) x w'(x) + \sigma^2\theta^* x^2 w''(x), \tag{9}$$

and

$$0 = u'(c^*) - w'(x), \tag{10}$$

which imply that the optimal policies must satisfy

$$\theta^*(x) = -\frac{w'(x)}{x w''(x)}\frac{(\mu - r)}{\sigma^2},$$

and

$$u'(c^*) = w'(x).$$

The natural candidate solution for $w(x)$ that corresponds to a CRRA, power instantaneous utility (4) is

$$w(x_s) = \alpha_H^{-b}\frac{x_s^{1-b}}{1-b}, \ \forall s \geq t, \tag{11}$$

and the corresponding optimal portfolio and consumption rules are [15]

$$\theta^* = \frac{1}{b}\frac{(\mu - r)}{\sigma^2}, \quad \forall x > 0, \tag{12}$$

and

$$c^*(x) = \alpha_H x, \tag{13}$$

where α_H stands for the *hyperbolic* marginal propensity to consume out of wealth. We note from (12) that the portfolio strategy is independent of the

[15] Note that equation (12) is obtained by inserting the derivatives of the candidate solution (11) into (9). In particular, the derivatives are $w'(x) = \alpha_H^{-b}x^{-b}$ and $w''(x) = -b\alpha_H^{-b}x^{-b}/x = -bw'(x)/x$.

current wealth level, and is *identical* to the portfolio rule of the CRRA, exponential, time-consistent agent studied in Merton (1969, 1971).[16] However, as we show below, the wealth dynamics will be affected by a different, *higher* marginal propensity to consume relative to the Merton case.

In order to find an explicit solution for α_H, we expand the terms in the right-hand side of the Bellman equation that characterizes the hyperbolic-discounting problem. First, we note that

$$u\left(c^*\left(x\right)\right) = \alpha_H w\left(x\right) \tag{14}$$

Then, inserting (12), (13) , (14) and the derivatives of the candidate solution $w'\left(x\right) = \left(1-b\right)\frac{1}{x}w\left(x\right)$, $w''\left(x\right) = -b\left(1-b\right)\frac{1}{x^2}w\left(x\right)$ into the Bellman equation (7), we obtain

$$\gamma w\left(x\right) = \alpha_H w\left(x\right) + \left(1-b\right)\left(-\alpha_H + r + \frac{\left(\mu-r\right)^2}{2b\sigma^2}\right)w\left(x\right) + \lambda\left(\beta v\left(x\right) - w\left(x\right)\right) \tag{15}$$

We show in the Appendix that the last term in the right-hand side of (15) is

$$\lambda\left(\beta v\left(x_t\right) - w\left(x_t\right)\right) = -\lambda\left(1-\beta\right)\alpha_H \int_t^\infty e^{-\left(\lambda+\gamma\right)\left(s-t\right)} E_t\left[w\left(x_s\right)\right] ds \tag{16}$$

where $E_t\left[w\left(x_s\right)\right]$ is determined by

$$E_t\left[\frac{w\left(x_s\right)}{w\left(x_t\right)}\right] = E_t\left[\left(\frac{x_s}{x_t}\right)^{1-b}\right] = \exp\left\{\left(1-b\right)\left(-\alpha_H + r + \frac{\left(\mu-r\right)^2}{2b\sigma^2}\right)\left(s-t\right)\right\}, \tag{17}$$

which is derived from the candidate solution (11) and the policy rules and by applying Ito's Lemma[17].

[16] Note that technically this is due to the fact that the present self has no control over the variables that determine the continuation value function $v\left(x\right)$, as can be verified in (8) .

[17] Specifically, the expected growth of wealth when the agent chooses optimal poli-

Finally, using (16) and (17) into (15), in account of the transversality condition **A3**, we obtain

$$\alpha_H = \frac{1}{b}\left(\underbrace{\left[\gamma + \alpha_H (1-\beta) \frac{\lambda}{\lambda + \gamma - (1-b)\left(-\alpha_H + \bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)}\right]}_{\textbf{Effective discount rate}} - (1-b)\left(\bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)\right) \tag{18}$$

where $\bar{\mu} \equiv \theta^* \mu + (1-\theta^*) r$ and $\bar{\sigma}^2 \equiv \theta^{*2}\sigma^2$ are the mean and variance of the optimal portfolio's rate of return[18]. The terms in brackets account for the *effective discount rate*, which is *greater* than γ under **A1** and **A2**. In the particular case where $\beta = 1$ or $\lambda = 0$, the effective discount rate would simply be γ and the marginal propensity to consume (MPC) out of wealth would correspond to the exponential discounting model treated in Merton (1969, 1971):

$$\alpha_M = \frac{1}{b}\left[\gamma - (1-b)\left(\bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)\right]. \tag{19}$$

In turn, for the case of interest where $\lambda \to \infty$ that corresponds to IG model, the MPC reduces to[19]

$$\alpha_{IG} \equiv \alpha_H|_{\lambda\to\infty} = \frac{1}{b-(1-\beta)}\left(\gamma - (1-b)\left(\bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)\right), \tag{20}$$

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cies (12)-(13) is given by $E\left[dx/x\right] = \left[-\alpha_H + r + \frac{1}{b}\frac{(\mu-r)^2}{\sigma^2}\right] dt$. Applying Ito's Lemma on the candidate function (11), the expected growth of the value function is $E\left[dw/w\right] = E\left[dx^{1-b}/x^{1-b}\right] = (1-b)\left(-\alpha_H + r + \frac{(\mu-r)^2}{2b\sigma^2}\right) dt$, which immediately implies (17). (See the Appendix for details on the derivations).

[18] Note that, the optimal policy $\theta^* = \frac{1}{b}\frac{(\mu-r)}{\sigma^2}$ given by (12) implies:

$$\bar{\mu} - b\frac{\bar{\sigma}^2}{2} = r + \frac{(\mu-r)^2}{2b\sigma^2} > 0.$$

[19] Note that if $\sigma = 0$, the Merton and the IG propensities to consume would reduce to: $\alpha_R = \alpha_M|_{\sigma=0} = \frac{1}{b}\left[\gamma - (1-b) r\right]$ and $\alpha_H|_{\substack{\lambda\to\infty \\ \sigma=0}} = \frac{1}{b-(1-\beta)}\left[\gamma - (1-b) r\right]$.

which is increasing in $\beta \in (0,1)$ and, under assumptions **A1-A2,** is unambiguously *greater* than the Merton's α_M.

Note that the MPC α_{IG} is *linear* in $\bar{\mu}$ and $\bar{\sigma}^2$ and is *not* wealth contingent.[20]

3 Discussion

The following results are obtained from the analysis:

i. Consumption, Savings and Portfolio Choices

First, the relative proportion of wealth allocated to stocks and bonds along the equilibrium path is *identical* to that obtained in the exponential discounting case (i.e. where $\beta = 1$ or $\lambda = 0$). This means that the size of the risk premium of stocks over bonds is also identical to that in the exponential case. In other words, the size of the equity premium is no more or less puzzling than what it is under exponential discounting.

Second, since the effective rate of time preference is greater than γ when $\beta \in (0,1)$ and $\lambda > 0$, the optimal marginal propensity to consume out of wealth α_H is unambiguously *greater* than the exponential discounting solution obtained by Merton (1969, 1971).[21] This is in line with the results in the literature that anticipate present bias.[22] Behind this conclusion is the assumption that the hyperbolic discounting model and the exponential discounting model have the same long-run discount rate. While this is a

[20] This is contrary to what has been posited in for example Gong et al. (2006, 2007).

[21] Note that in the particular case of *logarithmic* instantaneous utility, i.e. where $b \to 1$, for which the intertemporal subsitution effect and the wealth effect cancel each other, the marginal propensity to consume out of wealth is simply given by the subjective discount rate γ in the exponential model, and γ/β in the IG, hyperbolic model.

[22] See for instance O'Donoghue and Rabin (1999), Laibson (1994, 1997), Harris and Laibson (2003, 2008) and Luttmer and Mariotti (2003).

useful modelization typically followed in the literature that studies the implications of introducing dynamically inconsistent preferences, this needs not be the case.[23] For instance, the structural estimates reported in Laibson et al (2007) indicate that the β-δ quasi-hyperbolic model may have *more* short-run discounting and *less* long-run discounting than the exponential model. An alternative formulation could have followed this route and introduced two different parameters, rather than one, to compare the two models. We would then have reached the same general conclusion: hyperbolic discounting has quantitative implications for consumption-saving allocations and whether the model generates greater or lower consumption than in the exponential case depends on the specific parameters. For instance, the parameter estimates in Laibson et al (2007) indicate that hyperbolic discounting would indeed generate a greater consumption share.

Finally, note that under the feasibility assumption **A1**, α_H increases as $\beta \in (0,1)$ decreases and as $\lambda \in (0,\infty)$ increases. As stated above, the exponential discounting is a limit case where the marginal propensity to consume is $\alpha_M = \alpha_H|_{\beta=1} = \alpha_H|_{\lambda=0}$.

These results imply that outcomes are observationally equivalent to the exponential case with a suitably higher level of discounting. Barro (1999) also finds that the basic properties of the neoclassical growth model under exponential discounting remain intact when allowing for a variable rate of time preference.

ii. Level of Asset Prices and Returns

A lower saving rate than in the exponential case implies a lower demand for financial assets, which in turn implies lower prices and greater rates of return for both stocks and bonds. As a result, hyperbolic discounting preferences may predict a *greater* risk-free rate than exponential preferences as well

[23] We are grateful to a referee for pointing out this aspect.

as a *greater* return on equity. This implies that it would be easier to reconcile the size of the equity premium by simply explaining the size of the risk-free rate. In other words, under hyperbolic discounting there is less pressure to explain the size of returns on stocks and more pressure to explain the size of the risk-free rate than under a constant rate of time preference. In this sense, some of the potential solutions to the risk-free rate puzzle posited in the literature will have greater power to contributing to explain the size of the equity premium when discounting is hyperbolic rather than exponential.[24]

The IG case put forward in Harris and Laibson (2008), in which $\lambda \to \infty$, is of particular relevance for it has dealt with a number of problems inherent in the discrete-time approximation of hyperbolic discount functions, such as a kinked discount factor and the necessity to define the expected duration of the present term. Importantly, the IG model resolves the pathologies of multiplicity of equilibria and non-monotonicity of the consumption function that have flawed previous theoretical advances in the literature of time-inconsistent preferences. In particular, the properties of the IG model include the existence and uniqueness of equilibrium as well as the continuity and monotonicity of the consumption function.[25] For these reasons, in the discussions that follow we focus on this particular case of interest.

iii. Comparative Statics of Consumption with Respect to Risk: The Magnification Effect

Despite the fact that there are no implications for the risk premium other than quantitative implications for consumption-saving allocations and the level of asset returns, an important difference arises with regard to how consumption is related to risk.

Risk has a linear effect on consumption and depends on the degree of

[24] See, for instance, Kocherlakota (1996), Campbell (2000) and Mehra (2008) for reviews of the literature.

[25] See Harris and Laibson (2008, Section 5).

risk aversion. Merton (1969) calculated the elasticities of consumption with respect to expected return $\bar{\mu}$ and to variance $\bar{\sigma}^2$ for the exponential case, and noted that their sign depend on the parameter of risk aversion b :

$$\epsilon_{c,\bar{\mu}}|_M = \bar{\mu}\frac{b-1}{b}\frac{1}{\alpha_M},$$

and

$$\epsilon_{c,\bar{\sigma}^2}|_M = -\bar{\sigma}^2\frac{b-1}{2}\frac{1}{\alpha_M}.$$

In the hyperbolic IG case, the sign of the corresponding elasticities also depend on b being greater or lower than 1. However, the absolute value is unambiguously greater than in the Merton case for $b \neq 1$:[26]

$$\epsilon_{c,\bar{\mu}}|_{IG} = \bar{\mu}\frac{b-1}{b-(1-\beta)}\frac{1}{\alpha_H},$$

and

$$\epsilon_{c,\bar{\sigma}^2}|_{IG} = -\bar{\sigma}^2\frac{b}{b-(1-\beta)}\frac{b-1}{2}\frac{1}{\alpha_H}.$$

This result arises from the fact that the sensitivity of the hyperbolic MPC to changes in $\bar{\sigma}^2$ and $\bar{\mu}$ is greater than in the exponential case. In particular, in the exponential case the derivative of the MPC with respect to risk is

$$\frac{\partial \alpha_M}{\partial \bar{\sigma}^2} = \frac{1-b}{2}. \tag{21}$$

while in the IG model, the effect of risk on the marginal propensity to consume is

$$\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2} = \frac{b}{b-(1-\beta)}\frac{1-b}{2}. \tag{22}$$

We note that in both cases risk has a linear effect on the marginal propensity to consume, and that the derivatives are decreasing in risk for $b > 1$ and

[26] See Appendix for the derivation of the elasticities of consumption with respect to $\bar{\mu}$ and $\bar{\sigma}^2$.

increasing in risk for $b < 1$.[27] However, the absolute value of this relationship is greater in the hyperbolic case than in the exponential discounting case:

$$\left|\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2}\right| > \left|\frac{\partial \alpha_M}{\partial \bar{\sigma}^2}\right|, \qquad \forall b \neq 1.$$

This implies that for any $b \neq 1$, hyperbolic discounting generates a *greater* response of the propensity to consume to changes in risk.[28]

One implication is that although the hyperbolic discounting's MPC may well be observationally equivalent to the exponential case for given $\bar{\sigma}^2$, $\bar{\mu}$ and risk aversion parameter b (i.e. there would be a suitable subjective rate γ that would make $\alpha_{IG} = \alpha_M$), the MPC would react *more* to changes in the risk parameter. In models where risk is allowed to change (e.g. models with stochastic volatility), this result provides a magnification mechanism that would contribute to explaining the excess price volatility puzzle.

iv. Implications for Asset Prices in a Lucas Tree Model

We have seen that the relationship between the marginal propensity to consume and portfolio volatility is *magnified* through a greater absolute value of its slope for coefficients of relative risk aversion different than one. However, asset prices are exogenous in the Merton model. Hence, to determine the quantitative importance of hyperbolic discounting as a driving force behind stock market volatility, the introduction of a model with endogenous asset prices is in order. This can be done by considering a simple continuous-time version of Lucas' (1978) representative-agent fruit-tree model of asset pricing. A tree (stock) yields fruit (dividends) D_t according to a geometric Brownian motion:

$$\frac{dD_t}{D_t} = v\ dt + \sigma\ dZ_t.$$

[27] See Merton (1969), Section 7, for a discussion on the effect of changes in $\bar{\mu}$ and $\bar{\sigma}^2$ on consumption.

[28] Note that for logarithmic utility both slopes are equal to zero, i.e. $\frac{\partial \alpha_{IG}}{\partial \bar{\sigma}^2}\big|_{b=1} = \frac{\partial \alpha_M}{\partial \bar{\sigma}^2}\big|_{b=1} = 0$.

Investors can buy shares in the stock at price P_t. The supply of shares is normalized to 1 and we assume zero net supply of the risk-free asset. In equilibrium, the representative agent follows the optimal policy $c_t^* = \alpha^* x_t$, where α^* is the optimal MPC out of wealth. Ignoring bubble solutions, we conjecture that the equilibrium price is proportional to dividends. Since in equilibrium all dividends are consumed and wealth is equal to the value of the stock:

$$P_t = \frac{1}{\alpha^*} D_t.$$

The variability of prices will be directly linked to the variability of dividends and to the variablity of the MPC in specifications where, for example, the parameter σ were stochastic. As we indicated earlier, it can be shown that

$$\frac{\partial \alpha^*}{\partial \sigma^2} < 0 \text{ for } b > 1; \ \frac{\partial \alpha^*}{\partial \sigma^2} > 0 \text{ for } b < 1.$$

The intuition for this is the same as in the exponential case as established in Merton (1969). The slope of the schedule will depend on the relative strength of the substitution and income effects of the volatility parameter on consumption, which is determined by the risk aversion parameter b.[29] However, in the IG model the MPC and consequently prices are more responsive to changes in risk.

The implications of these findings are the following:

1. From a theoretical perspective, these drastic differences in the comparative statics of consumption and asset prices with respect to risk mean that hyperbolic discounting offers a novel mechanism whereby changes in risk may affect consumption and stock prices.

2. From an empirical perspective, in order to get a sense of the possible quantitative size of the volatility effects, it is useful to study a calibrated model in the region of the parameter space that is empirically plausi-

[29] See Merton (1969), Section 7, for a detailed discussion.

ble. Laibson et al. (2007) use a structural model and field data to estimate an unrestricted discount function that allows the discount rate to differ in the short-run and in the long-run. Theirs are, to the best of our knowledge, the best available estimates obtained in field data. Their structural procedure yields estimates for their benchmark case (which sets the relative risk aversion parameter at a value of 2) of $\beta = 0.7031$ and $\delta = 0.9580$, with standard errors of 0.1093 and 0.0068 respectively. Letting, for instance, the relative risk aversion parameter be 3, yields an estimate of $\beta = 0.5776$ with a standard error of 0.1339, leaving basically unchanged the estimate of δ.

Since they consider a rich consumption model that includes stochastic labor income, liquid and illiquid assets, revolving credit and other ingredients, they can also perform several robustness tests to changes in the different parameters of the model. These tests include compound cases where parameter changes are allowed to reinforce each other. The evidence they obtain indicates that a reasonable range for the parameters we are interested in is for the parameter β from 0.40 to 0.80 and for the paramenter b from 1 to 3.[30]

In Table 1 below we report the results of the calibrations for different parameter values of the ratio:

$$\Omega = \left.\frac{\partial \alpha_{IG}}{\partial \sigma^2}\right|_{\beta<1} / \left.\frac{\partial \alpha_M}{\partial \sigma^2}\right|_{\beta=1} = \frac{b}{b-(1-\beta)}$$

which captures the *magnification* in the response of the MPC to changes in risk relative to the standard, exponential-discounting case. We explore different combinations of the parameter of relative risk aversion b and the short-run discount factor β.

[30] As Laibson et al. (2007) discuss, the picture with respect to the relative risk aversion coefficient is not entirely clear. They consider a value of 2 for their benchmark case, and also the values 1 and 3. The usual view in the asset pricing and consumption-savings literatures is that it is in the range of 0.5 to 5. Gourinchas and Parker (2002) find values between 0.1 and 5.3. Barro (2006) notes that savings rates (excluding human capital) fall markedly as a country develops if b is much below 2, and are counterfactually low it is much above 4.

[Table 1 about here]

We find that Ω increases with β, for a given value of b, and similarly, given β, Ω increases with b. In particular, for the preferred estimate in Laibson et al (2007) of $\beta = 0.70$, we find that hyperbolic discounting generates between 11.1 percent (for $b = 1$) to 42.8 percent (for $b = 3$) greater responsiveness of the MPC to changes in risk than in the standard formulation.

For the case of $\beta = 0.80$, which is a value that might seem to be on the high end, hyperbolic discounting generates between a 7.1 to 25 percent greater response of prices to changes in risk than exponential discounting, whereas for $\beta = 0.40$, a value which should not be considered unrealistically low given the findings in Laibson et al (2007), we find that hyperbolic discounting generates a magnification of the response of the MPC to changes in risk that is between 25 to 150 percent greater than in the exponential case. .

Summing up, calibrations of the Lucas model with hyperbolic discounting in the empirically plausible region of the parameter space reveal that the MPC, and therefore prices, are much more responsive to changes in risk than in the standard case. Most of the calibrations indicate that this responsiveness is typically in the range of 10 to 50 percent greater, except when both β and b are low when the responsiveness is above 50 percent greater.

We conclude that in light of the difficulties in the literature for explaining stock market volatility, also known as the excess-volatility puzzle for stocks, hyperbolic discounting offers a great deal of promise for contributing to explaining an important and challenging aspect of asset market data.

4 Concluding Comments

The analysis has introduced dynamically inconsistent preferences in the standard setting where capital markets are perfect and frictionless, and where wealth is generated by stochastic returns on assets. Introducing labor income jointly with the constraint that consumers may not borrow against future labor income, incomplete markets, and other market frictions (e.g., taxes, transaction costs) are directions that merit future research, even though it is typically not possible to obtain closed-form solutions in these settings.

Over the last couple of decades a large literature has significantly departed from the tractable framework of a representative-agent, time-additive isoelastic preferences, and complete frictionless markets in an attempt to explain asset pricing puzzles. This paper maintains the assumption of time and state-separable preferences defined only over consumption.

Since the analysis considers a frictionless economy with no liquidity constraints, it becomes readily comparable to Merton (1969, 1971) and differentiates our work from Laibson (1994, 1997) and Harris and Laibson (2003, 2008). As in Merton's setup with constant relative risk aversion, the porfolio-selection decision is independent of the consumption decision. We provide closed-form solutions for a hyperbolic agent's optimal consumption and the optimal porfolio-selection problems and show that the latter is identical to the Merton's exponential case.

As for consumption, we obtain a closed-form solution that shows that the marginal propensity to consume is pinned down from a system of ordinary differential equations. We show that the hyperbolic MPC is unambiguously greater compared to the classic exponential case and, in addition, how the MPC is more sensitive to changes in the risk and expected return parameters. These findings suggest that this model of time-inconsistent preferences has potential to contribute to explaining challenging aspects of asset market data, particularly of stock market volatility. We leave this as a recommendation for further research.

References

[1] Ainslie, G. and Haslam, N. (1992). "Hyperbolic Discounting". In G. Loewenstein and J. Elster, eds., *Choice Over Time*. Russell Sage Foundation, pp. 57-92.

[2] Barro, R. (2006). "Rare Disasters and Asset Markets in the Twentieth Century." *Quarterly Journal of Economics* 121, pp. 823-866.

[3] Barro, R. (1999). "Ramsey Meets Laibson in the Neoclassical Growth Model." *Quarterly Journal of Economics* 114, pp. 1125-1152.

[4] Campbell, J. Y. (2000). "Asset Pricing at the Millennium." *Journal of Finance* 55; pp. 1515-1567.

[5] Della Vigna, S. and U. Malmendier. (2004)."Contract Design and Self-Control: Theory and Evidence." *Quarterly Journal of Economics* 119, pp. 353-402.

[6] Gong, L., W. Smith, and H. Zou (2006). "Consumption and Risk with Hyperbolic Discounting." *Economics Letters* 96, pp. 153-160.

[7] Gong, L., W. Smith, and H. Zou (2007). "Asset Prices and Hyperbolic Discounting." *Annals of Economics and Finance* 8-2, pp. 397–414

[8] Gourinchas, P.-O. and J. Parker (2002)."Consumption over the Life-Cycle." *Econometrica* 70, pp. 47-89.

[9] Gul, F. and W. Pesendorfer (2001)."Temptation and Self-Control." *Econometrica* 69, pp. 1403-1435.

[10] Gul, F. and W. Pesendorfer (2004). "Self-Control and the Theory of Consumption." *Econometrica* 72, pp. 119-158.

[11] Halevy, Y. (2008). "Strotz Meets Allais: Diminishing Impatience and the Certainty Effect." *American Economic Review* 98 (3), pp. 1145-1162.

[12] Harris, C. and D. I. Laibson (2003). "Hyperbolic Discounting and Consumption." In *Advances in Economics and Econometrics, Theory and Applications*. Eighth World Congress. Edited by Dewatripont, Hansen and Turnovsky, vol. I, chapter 7.

[13] Harris, C. and D. I. Laibson (2001)."Instantaneous Gratification." Mimeo, Department of Economics, Harvard University.

[14] Harris, C. and D. I. Laibson (2008). "Instantaneous Gratification." Mimeo, Department of Economics, Harvard University.

[15] Herrnstein, R. J. (1997). *The Matching Law.* Edited by Howard Rachlin and David I. Laibson. Cambridge, MA: Harvard University Press.

[16] Hirshleifer, D. (2001). "Investor Psychology and Asset Pricing." *Journal of Finance* 56; pp. 1533-1597.

[17] Kocherlakota, N. R. (1996). "The Equity Premium: It's Still a Puzzle." *Journal of Economic Literature* 34, pp. 42-71.

[18] Krussell, P., Kurusçu, B. and A. Smith Jr. (2002). "Time Orientation and Asset Prices." *Journal of Monetary Economics* 49, pp. 107-135.

[19] Krussell, P. and A. Smith Jr. (2003). "Consumption-Savings Decision with Quasi-Geometric Discounting." *Econometrica* 71, pp. 365-375.

[20] Laibson, D. I. (1994). "Hyperbolic Discounting and Consumption". Ph. D. Dissertation, Massachusetts Institute of Technology.

[21] Laibson, D. I. (1997). "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics* 112, pp. 443-477.

[22] Laibson, D. I., A. Repetto, and J. Tobacman (2007). "Estimating Discount Functions with Consumption Choices over the Lifecycle." Mimeo, Harvard University.

[23] Lucas, R. E. Jr (1978). "Asset Prices in an Exchange Economy," *Econometrica* 46, pp. 1429-1445.

[24] Luttmer, E. G. J. and T. Mariotti (2003). "Subjective Discounting in an Exchange Economy." *Journal of Political Economy* 111, pp. 959-989.

[25] McClure, S. M., D. Laibson, G. Loewenstein and J. D. Cohen (2004). "Separate Neural Systems Value Immediate and Delayed Monetary Rewards." *Science* 306, October 15.

[26] Mehra, R. (2008). "The Equity Premium Puzzle: A Review." *Foundations and Trends in Finance*, vol.2, no.1, pp. 1-81.

[27] Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case." *Review of Economics and Statistics* 51, pp. 247-257.

[28] Merton, R. C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory* 3, pp. 373-413.

[29] O'Donoghue, T. and M. Rabin. (1999). "Doing It Now or Later." *American Economic Review* 89(1), pp. 103-124.

[30] Strotz, R. H. (1955). "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies* 23, pp. 165-180.

[31] Sundaresan, S. M. (2000). "Continuous-Time Methods in Finance: A Review and an Assessment." *Journal of Finance* 55, pp. 1569-1622.

5 Appendix

5.1 Derivation of Bellman equation (7)

The objective function can be written as in (6)

$$w(x) = u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[e^{-\gamma dt} \beta v(x+dx)]$$

Multiply both sides by $e^{\gamma dt}$ to get

$$e^{\gamma dt} w(x) = e^{\gamma dt} u(c(x))\,dt + e^{-\lambda dt} E[e^{-\gamma dt} w(x+dx)] + \left(1 - e^{-\lambda dt}\right) E[\beta v(x+dx)]$$

For small dt, we can approximate

$$\begin{aligned} e^{-\lambda dt} &\approx 1 - \lambda dt \\ e^{-\gamma dt} &\approx 1 - \gamma dt \\ e^{\gamma dt} &\approx 1 + \gamma dt \end{aligned}$$

Therefore, the above equation can be written as

$$(1 + \gamma dt)\, w(x) \approx (1 + \gamma dt)\, u(c(x))\,dt + (1 - \lambda dt)\, E[w(x+dx)] + \lambda dt E[\beta v(x+dx)]$$

Substracting $w(x)$ from both sides we get

$$\gamma dt w(x) \approx (1 + \gamma dt)\, u(c(x))\,dt + (1 - \lambda dt)\, E[dw(x)] + \lambda dt E[\beta v(x+dx) - \beta v(x) - w(x) + \beta v(x)]$$

where

$$\begin{aligned} dw(x) &= w(x+dx) - w(x), \\ dv(x) &= v(x+dx) - v(x) \end{aligned}$$

Dividing by dt

$$\gamma w(x) \approx (1 + \gamma dt)\, u(c(x)) + (1 - \lambda dt)\, \frac{E[dw(x)]}{dt} + \lambda E[\beta dv(x) - w(x) + \beta v(x)]$$

and letting $dt \to 0$, we obtain

$$\gamma w(x) = u(c(x)) + \frac{E[dw(x)]}{dt} + \lambda E[\beta v(x) - w(x)]. \tag{23}$$

Applying Ito's Lemma and taking expectations we find

$$E\left[dw\left(x\right)\right] = w'E\left[dx\right] + \frac{1}{2}w''E\left[\left(dx\right)^2\right]$$

where, from (2),

$$E\left[dx\right] = \left(\mu\theta x + \left(1-\theta\right)r\theta x - c\right)dt$$

and

$$E\left[\left(dx\right)^2\right] = \sigma^2\theta^2x^2dt$$

Thus, the Bellman equation (23) can be written as

$$\gamma w\left(x\right) = u\left(c\left(x\right)\right) + \left(\mu\theta x + \left(1-\theta\right)r\theta x - c\right)w'\left(x\right) + \frac{1}{2}\sigma^2\theta^2x^2w''\left(x\right) + \lambda\left(\beta v\left(x\right) - w\left(x\right)\right).$$

Q.E.D.

5.2 Derivation of (16)

Recall the definitions

$$w\left(x_t\right) = E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds + \int_{t+\tau_t}^{\infty}\beta e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds\right]$$

and

$$\begin{aligned} v\left(x_t\right) &\equiv E_t\int_t^{\infty} e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds \\ &= E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds + \int_{t+\tau_t}^{\infty} e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds\right] \end{aligned}$$

or

$$\beta v\left(x_t\right) = E_t\left[\int_t^{t+\tau_t} \beta e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds + \int_{t+\tau_t}^{\infty}\beta e^{-\gamma(s-t)}u\left(c\left(x_s\right)\right)ds\right]$$

Therefore,

$$\begin{aligned}
\beta v(x_t) - w(x_t) &= E_t\left[\int_t^{t+\tau_t} \beta e^{-\gamma(s-t)} u(c(x_s))\, ds - \int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds\right] \\
&= -(1-\beta) E_t\left[\int_t^{t+\tau_t} e^{-\gamma(s-t)} u(c(x_s))\, ds\right] \\
&= -(1-\beta) \int_t^{\infty} e^{-\lambda(s-t)} e^{-\gamma(s-t)} E_t[u(c(x_s))]\, ds \\
&= -(1-\beta) \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t[u(c(x_s))]\, ds
\end{aligned}$$

So,

$$\lambda(\beta v(x_t) - w(x_t)) = -\lambda(1-\beta) \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t[u(c(x_s))]\, ds.$$

or, using (14),

$$\lambda(\beta v(x_t) - w(x_t)) = -\lambda(1-\beta)\alpha_H \int_t^{\infty} e^{-(\lambda+\gamma)(s-t)} E_t[w(x_s)]\, ds$$

Q.E.D.

5.3 Derivation of (17)

This equation can be derived making use of the candidate solution (11) and the policy rules. First, recall the candidate value function (11) and apply Ito's Lemma to find $dw = w'dx + \frac{1}{2}w''(dx)^2$. Recall that the variation dx is given by (2), and note that $(dx)^2 = \sigma^2\theta^2x^2dt$. Thus,

$$dw = w'dx + \frac{1}{2}w''(dx)^2$$

$$dw = \left([rx + (\mu - r)\theta x - c]w' + \frac{1}{2}\sigma^2\theta^2x^2w''\right)dt + \sigma\theta x w' dz$$

so the expected variation of the value function is

$$E[dw] = \left([rx + (\mu - r)\theta x - c]w' + \frac{1}{2}\sigma^2\theta^2x^2w''\right)dt$$

where the partial derivatives are $w' = (1-b)\frac{w}{x}$, and $w'' = -b(1-b)\frac{w}{x^2}$. It follows from (12) and (13) that the variation of the *optimal* value function is

$$dw = (1-b)\left(-\alpha_H + r + \frac{(\mu - r)^2}{2b\sigma^2}\right)wdt + \frac{(1-b)}{b}\frac{(\mu - r)}{\sigma}wdz,$$

with expectation

$$E[dw] = (1-b)\left(-\alpha_H + r + \frac{(\mu - r)^2}{2b\sigma^2}\right)wdt$$

which implies

$$E_t[w_s] = w(x_t)\exp\left\{(1-b)\left(-\alpha_H + r + \frac{(\mu - r)^2}{2b\sigma^2}\right)(s-t)\right\}.$$

Q.E.D.

Alternatively we could recall the candidate solution (11)

$$w(x_s) = \alpha_H^{-b}\frac{x_s^{1-b}}{1-b},\ \forall s \geq t$$

and that $w' = (1-b)\frac{w}{x}$, and $w'' = -b(1-b)\frac{w}{x^2}$. And from Ito's Lemma:

$$dw = w'dx + \frac{1}{2}w''(dx)^2$$

5.4 Derivation of elasticities of consumption with respect to portfolio's expected return and variance

Recall the MPC of the hyperbolic IG model (20)

$$\alpha^{IG} \equiv \alpha_H|_{\lambda\to\infty} = \frac{1}{b-(1-\beta)}\left(\gamma - (1-b)\left(\bar{\mu} - b\frac{\bar{\sigma}^2}{2}\right)\right).$$

The elasticity of consumption with respect to expected returns would be

$$\epsilon_{c,\bar{\mu}}|_{IG} \equiv \bar{\mu}\frac{\partial c}{\partial\bar{\mu}}\frac{1}{c}$$

Taking the derivative

$$\frac{\partial c}{\partial \bar{\mu}} = \frac{\partial \alpha}{\partial \bar{\mu}} x + \alpha \frac{\partial x}{\partial \bar{\mu}}$$

we note that $\frac{\partial x_0}{\partial \bar{\mu}} = 0$, and

$$\frac{\partial c}{\partial \bar{\mu}} = \frac{\partial \alpha}{\partial \bar{\mu}} x = \frac{b-1}{b-(1-\beta)} x$$

therefore

$$\epsilon_{c,\bar{\mu}}|_{IG} = \bar{\mu} \frac{b-1}{b-(1-\beta)} \frac{x}{c}$$

Recalling (13) , we can write

$$\epsilon_{c,\bar{\mu}}|_{IG} = \bar{\mu} \frac{b-1}{b-(1-\beta)} \frac{1}{\alpha}$$

In similar way we can find the elasticity of consumption with respect to portfolio volatility

$$\epsilon_{c,\bar{\sigma}^2}|_{IG} = -\bar{\sigma}^2 \frac{b}{b-(1-\beta)} \frac{b-1}{2} \frac{1}{\alpha_H}.$$

TABLE 1 - RESPONSE OF STOCK PRICES TO CHANGES IN RISK UNDER HYPERBOLIC DISCOUNTING RELATIVE TO EXPONENTIAL DISCOUNTING

This table reports the ratio $\Omega = (\partial\alpha_{IG}/\partial\sigma^2 \big|_{\beta<1}) / (\partial\alpha_M/\partial\sigma^2 \big|_{\beta=1}) = b / (b-(1-\beta))$ for various combinations of the relative risk aversion parameter b and the short-run discount factor β.

Relative Risk Aversion b	Short-run discount factor β						
	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$
1.00	2.500	2.000	1.666	1.428	1.250	1.111	1.000
1.20	2.000	1.714	1.500	1.333	1.200	1.090	1.000
1.40	1.750	1.556	1.400	1.272	1.167	1.076	1.000
1.60	1.600	1.455	1.333	1.231	1.143	1.066	1.000
1.80	1.500	1.385	1.286	1.200	1.125	1.058	1.000
2.00	1.429	1.333	1.250	1.176	1.111	1.052	1.000
2.20	1.375	1.294	1.222	1.158	1.100	1.047	1.000
2.40	1.333	1.263	1.200	1.143	1.091	1.043	1.000
2.60	1.300	1.238	1.182	1.130	1.083	1.040	1.000
2.80	1.273	1.217	1.167	1.120	1.077	1.037	1.000
3.00	1.250	1.200	1.154	1.111	1.071	1.034	1.000